

Periodic Rigidity on a Variable Torus Using Inductive Constructions

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Abstract

In this paper we prove a recursive characterisation of generic rigidity for frameworks periodic with respect to a partially variable lattice. We follow the approach of modelling periodic frameworks as frameworks on a torus and use the language of gain graphs for the finite counterpart of a periodic graph. In this setting we employ variants of the Henneberg operations used frequently in rigidity theory.

1 Introduction

Given an embedding of a graph into Euclidean space as a collection of stiff bars and flexible joints (a framework), when is it possible to continuously deform the structure into a non-congruent position without breaking connectivity or changing the bar lengths (is the framework rigid or flexible)? This is the fundamental question in rigidity theory. The subject has many obvious applications, for example, in molecular biology, structural engineering and computer aided design [7].

Typically the question is *NP*-hard [1] but for generic embeddings in the plane there is a complete combinatorial description.

Theorem 1.1 (Henneberg [10], Laman [12], Maxwell [15]). *Let $G = (V, E)$ and let p be a generic embedding into \mathbb{R}^2 . Then the following are equivalent:*

- (1) *the framework (G, p) is generically minimally rigid,*
- (2) *G satisfies $|E| = 2|V| - 3$ and $|E'| \leq 2|V'| - 3$ for every $G' = (V', E') \subset G$,*
- (3) *G can be constructed from a single edge by recursively adding vertices of degree 2 and by removing an edge and adding vertices of degree 3 adjacent to the ends of the old edge.*

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We will assume some basic familiarity with the rigidity of finite frameworks, those unfamiliar may wish to consult [6], [7] or [26]. The operations defined in (3) will be referred to as Henneberg operations, [10].

Recently a lot of attention has been paid to finding analogues of Theorem 1.1 for infinite graphs, specifically for periodic graphs (those with a finite quotient in an appropriate sense). Particularly [19] and [20] developed the approach of considering periodic frameworks as frameworks embedded on a torus. Indeed in [20] a characterisation was given for the case where the lattice (or torus) is fixed.

The purpose of this paper is to prove the following result concerning the case when one of the lattice vectors is allowed to change.

Theorem 1.2. *A labelled graph is generically minimally rigid on the partially variable torus if and only if it can be derived from a single loop by gain-preserving Henneberg operations.*

The gain-preserving Henneberg operations are extensions of the recursive moves in Theorem 1.1 (3) to include labelled edges, and will be formally defined in Section 4. One of the intricacies of the theorem is that we are working with a class of graphs, the $P(2,1)$ -graphs (defined in Section 3), that fall in between $(2,1)$ -tight graphs and $(2,2)$ -circuits. It is, in part, for this reason that we do not adopt a matroid theoretic approach in this paper.

Throughout the paper, we focus on a particular type of variable torus, namely one which is variable in the x -direction only. In fact, the results are much more general, and apply to frameworks on a torus which is variable in the y -direction only, and frameworks on a torus which has a variable angle between two fixed-length generators. We may also apply this result for a full characterisation of the generic rigidity of frieze-type patterns, that is, frameworks which are periodic in one direction only (interpreted as frameworks on a variable cylinder). These variations are discussed in Section 7.

The study of periodic frameworks is a topic which has experienced a surge of interest over the past decade [3, 4, 14, 18, 9]. This work has been motivated in part by questions arising in materials science about the structural properties of *zeolites*, a type of mineral with a repetitive, micro-porous structure [22]. Furthermore, there may be physical meaning associated with certain restrictions of the fully variable torus. It has been suggested that the time scales of atomic movement are significantly different from those of lattice deformation [23].

In this paper we take an inductive approach to the problem of characterizing the generic rigidity of periodic frameworks. That is, we define a collection of local graph-theoretic moves which characterize the class of generically rigid periodic frameworks on a partially variable torus. The inductive method has the advantage of being easy to state and understand. Furthermore, while finding an inductive construction for a particular graph does not in general make for fast algorithms, once we have such an inductive sequence, it offers an immediate certificate of the rigidity of that framework.

1.1 Results in Context

The basic theory of periodic frameworks has been well formalized by Borcea and Streinu [4]. The approach we use here is based on the presentation appearing in [19]. In that paper, the links between the approach of [4] and the present methodology are outlined in detail.

In [20], Ross proved an inductive characterisation of the generic rigidity of two-dimensional periodic frameworks on a *fixed torus*, that is, a torus with no variability. The methods used here build on those results. In [14] Malestein and Theran proved a characterisation of generic minimal rigidity of two-dimensional frameworks on the fully variable torus (three degrees of freedom). They obtain the result of Ross as a restriction of their more general theorem. However, their methods differ significantly from ours, in that they do not use an inductive characterisation. Indeed giving an inductive construction for the relevant class of graphs on the fully variable torus is an intriguing open problem.

1.2 Outline of Paper

In Section 2 we recall the basic theory of periodic frameworks as frameworks on a torus. The following two sections state the relevant rigidity results for the partially variable torus, Maxwell-type necessary conditions and Henneberg constructions preserving rigidity of frameworks. In Section 5 we prove some preliminary graph theory results, including an inductive construction of $P(2, 1)$ -graphs that may be of independent interest. The main body of the paper is contained in Section 6 where we present a case by case analysis showing that the appropriate gains are preserved by the construction operations. This allows us to prove our main theoretical result, Theorem 6.1, and hence to complete the proof of Theorem 1.2. We describe some extensions of the work in Section 7. The final section concludes the paper with some discussion of further work.

2 Background

A *periodic framework* in the plane is a locally finite infinite graph which is symmetric with respect to the free action of \mathbb{Z}^2 . Such a framework has a finite number of vertex and edge orbits under the action of \mathbb{Z}^2 . Full definitions and details can be found in the work of Borcea and Streinu, [3, 4]. The approach taken here is to consider periodic frameworks as orbit frameworks on a torus, as in [19, 20].

2.1 Periodic Orbit Frameworks on the Variable Torus \mathcal{T}_x^2

Let $\mathcal{T}_x^2 = \mathbb{R}^2 / L_x \mathbb{Z}^2$, where

$$L_x = L_x(t) = \begin{pmatrix} x(t) & 0 \\ y_1 & y_2 \end{pmatrix}, \quad y_1, y_2 \in \mathbb{R}.$$

We call \mathcal{T}_x^2 the *x-variable torus*, and the matrix $L_x = L_x(t)$ is the lattice matrix. Similarly, let

$$L_0 = \begin{pmatrix} x & 0 \\ y_1 & y_2 \end{pmatrix}, \quad x, y_1, y_2 \in \mathbb{R}$$

be the *fixed lattice matrix*, and we call the quotient space $\mathcal{T}_0^2 = \mathbb{R}^2/L_0\mathbb{Z}^2$ the *fixed torus*.

For a graph G , we will use the notation $V(G)$ and $E(G)$ to refer to the vertex and edge sets of G , if not explicitly named. A *periodic orbit framework* $(\langle G, m \rangle, p)$ on \mathcal{T}_x^2 consists of a labelled, directed multigraph $\langle G, m \rangle$ together with a position p of the vertices $V(G)$ on the variable torus \mathcal{T}_x^2 . $\langle G, m \rangle$ is called a *gain graph* [28, 8], and it encodes the way in which the graph G is “wrapped” around the torus. Specifically, $\langle G, m \rangle$ is composed of a directed multigraph $G = (V, E)$, and a labelling $m : E^+ \rightarrow \mathbb{Z}^2$. The labelling of the edges is invertible, meaning that we may write

$$e = \{v_i, v_j; m_e\} = \{v_j, v_i; -m_e\}.$$

From the periodic orbit graph we may define the derived periodic graph G^m , which has vertex set $V^m = V \times \mathbb{Z}^2$, and edge set $E^m = E \times \mathbb{Z}^2$. If $e = \{v_i, v_j; m_e\} \in E\langle G, m \rangle$, then the edge $(e, z) \in E^m, z \in \mathbb{Z}^2$ connects the vertices (v_i, z) and $(v_j, z + m_e) \in V^m$. In this way, the periodic orbit graph $\langle G, m \rangle$ is a kind of “recipe” for the infinite derived graph G^m (see Figure 1). Furthermore, the automorphism group of G^m contains \mathbb{Z}^2 .

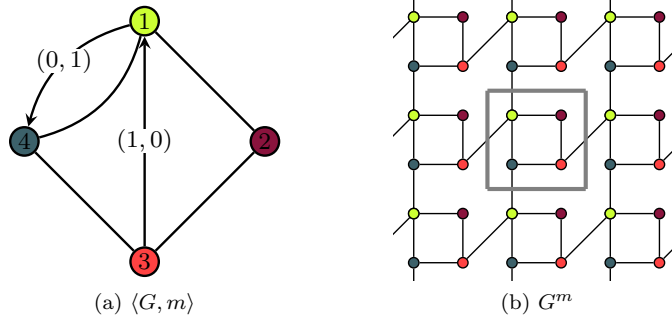


Figure 1: The periodic orbit graph $\langle G, m \rangle$ and its corresponding derived graph G^m .

In a similar fashion we use the periodic orbit framework $(\langle G, m \rangle, p)$ to define an infinite periodic framework, the *derived periodic framework* $(\langle G^m, L_x \rangle, p^m)$, where $p^m : V^m \rightarrow \mathbb{Z}^2$ is given by

$$p^m(v, z) = p(v) + zL_x, \quad \text{where } v \in V, z \in \mathbb{Z}^2.$$

An *infinitesimal motion* of $(\langle G, m \rangle, p)$ on \mathcal{T}_x^2 is an element $(u, u_x) \in \mathbb{R}^{2|V|+1}$, where

$$u : V \rightarrow \mathbb{R}^2, \quad \text{and } u_x : x(t) \rightarrow \mathbb{R},$$

such that

$$(p_i - p_j - m_e L_x) \cdot (u_i - u_j - (u_x(x), 0)) = 0 \text{ for all } \{v_i, v_j; m_e\} \in E\langle G, m \rangle. \quad (1)$$

If $u_x = 0$, and $u : V \rightarrow z, z \in \mathbb{R}^d$ (i.e. u is a translation), then we say that (u, u_x) is a trivial motion of $(\langle G, m \rangle, p)$ on \mathcal{T}_x^2 . If the only infinitesimal motions of a framework $(\langle G, m \rangle, p)$ on \mathcal{T}_x^2 are trivial, then we say that $(\langle G, m \rangle, p)$ is *infinitesimally rigid* on \mathcal{T}_x^2 .

Any infinitesimal motion (u, u_x) of $(\langle G, m \rangle, p)$ on the x -variable torus \mathcal{T}_x^2 for which $u_x = 0$ is also an infinitesimal motion of $(\langle G, m \rangle, p)$ on the fixed torus \mathcal{T}_0^2 . The definitions of trivial motions and infinitesimal rigidity on \mathcal{T}_0^2 are the same as for \mathcal{T}_x^2 .

2.2 The Rigidity Matrix \mathbf{R}_x

The *rigidity matrix* \mathbf{R}_x permits us to simultaneously solve the equations (1) for the space of infinitesimal motions of $(\langle G, m \rangle, p)$. It is an $|E| \times (2|V| + 1)$ matrix with one row corresponding to each edge, two columns corresponding to each vertex, and a single column corresponding to the variable lattice element $x(t)$. The row corresponding to the edge $\{v_i, v_j; m_e\}$ is as follows:

$$\begin{array}{ccccccc} & i & & j & & & x(t) \\ (0 \cdots 0 & p_i - (p_j + m_e L_x) & 0 \cdots 0 & (p_j + m_e L_x) - p_i & 0 \cdots 0 & (m_e)_x [p_i - (p_j + m_e L_x)]_x), \end{array}$$

where the entries under i and j are actually 2-tuples. By $(m_e)_x$ we mean the x -component of $m_e \in \mathbb{Z}^2$. The kernel of this matrix is the space of infinitesimal motions of $(\langle G, m \rangle, p)$ on \mathcal{T}_x^2 , and we may write

$$\mathbf{R}_x(\langle G, m \rangle, p) \cdot (u, u_x)^T = 0,$$

where $(u, u_x) \in \mathbb{R}^{2|V|+1}$ is as described above.

A framework on \mathcal{T}_x^2 always has a two-dimensional space of trivial infinitesimal motions, generated by the unit translations. It follows that the kernel of the rigidity matrix always has dimension at least 2. Furthermore, since a framework is infinitesimally rigid on \mathcal{T}_x^2 if and only if the only infinitesimal motions are trivial (i.e. are translations), we have the following result:

Theorem 2.1. *A periodic orbit framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on the x -variable torus \mathcal{T}_x^2 if and only if the rigidity matrix $\mathbf{R}_x(\langle G, m \rangle, p)$ has rank $2|V| - 1$.*

2.3 The T -gain Procedure on \mathcal{T}_x^2

In [19] Ross described the *T-gain procedure*, and showed that the rigidity matrices corresponding to two T -gain equivalent periodic orbit frameworks have the same rank. We now extend this to the variable torus case. See Figure 2 for an example.

The *net gain* on a cycle in a periodic orbit graph $\langle G, m \rangle$ is the sum of the gains on the edges of an oriented cycle of G , where the gains are appropriately multiplied by ± 1 , depending on the direction of traversal. The T -gain procedure is a procedure that can be used to easily identify the net gains on the cycles of a periodic orbit graph $\langle G, m \rangle$. As we will soon see, the rigidity of frameworks on \mathcal{T}_x^2 is generically characterized by the net gains on the cycles of the periodic orbit graph (Theorem 1.2). The T -gain procedure will thus be an essential proof technique which we use to show the necessity of the conditions in our main result (Proposition 3.1). The T -gain procedure is defined in [8] for general gain (voltage) graphs.

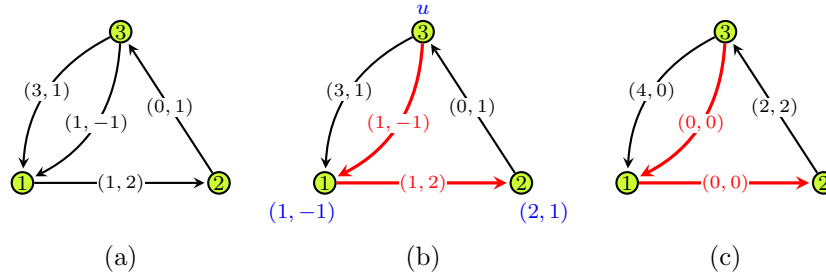


Figure 2: A gain graph $\langle G, m \rangle$ in (a), with identified tree T (in red), root u , and T -potentials in (b). The resulting T -gain graph $\langle G, m_T \rangle$ is shown in (c).

T -gain Procedure

1. Select an arbitrary spanning tree T of G , and choose a vertex u to be the root vertex.
2. For every vertex v in G , there is a unique path in the tree T from the root u to v . Denote the net gain along that path by $m(v, T)$, and we call this the T -potential of v . Compute the T -potential of every vertex v of G .
3. Let e be a plus-directed edge of G with initial vertex v and terminal vertex w . Define the T -gain of e , $m_T(e)$ to be

$$m_T(e) = m(v, T) + m(e) - m(w, T).$$

Compute the T -gain of every edge in G . Note that the T -gain of every edge of the spanning tree will be zero.

Theorem 2.2. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework on \mathcal{T}_x^2 . Then $\text{rank} \mathbf{R}_x(\langle G, m \rangle, p) = \text{rank} \mathbf{R}_x(\langle G, m_T \rangle, p')$, where $p' : V \rightarrow \mathbb{R}^2$ is given by $p'_i = p_i + m_T(v_i)$.*

Proof. Suppose that a set of rows is dependent in $\mathbf{R}_x(\langle G, m \rangle, p)$. Then there exists a vector of scalars, say $\omega = [\omega_1 \ \cdots \ \omega_{|E|}]$ such that

$$\omega \cdot \mathbf{R}_x(\langle G, m \rangle, p) = 0.$$

For each vertex $v_i \in V$ the column sum of $\mathbf{R}_x(\langle G, m \rangle, p)$ becomes

$$\sum_{e_\alpha \in E_+} \omega_{e_\alpha} (p_i - (p_j + m_{e_\alpha})) + \sum_{e_\beta \in E_-} \omega_{e_\beta} (p_i - (p_k - m_{e_\beta})) = 0, \quad (2)$$

where E_+ and E_- are the edges directed out from and into vertex i respectively. In [19], it was demonstrated that (2) is equivalent to the following:

$$\begin{aligned} \sum_{e_\alpha \in E_+} \omega_e \left(p_i + m_T(v_i) - (p_j + m_T(v_j)) - m_T(e) \right) + \\ \sum_{e_\beta \in E_-} \omega_e \left(p_i + m_T(v_i) - (p_j + m_T(v_j)) + m_T(e) \right) = 0 \end{aligned} \quad (3)$$

which is the column sum of the column of $\mathbf{R}_x(\langle G, m_T \rangle, p')$ corresponding to the vertex v_i .

Since we are working with the variable torus, we have one additional column corresponding to the flexibility of the x -direction. We will show that if there exists a vector of scalars $\omega = [\omega_1 \ \cdots \ \omega_{|E|}]$ such that

$$\omega \cdot \mathbf{R}_x(\langle G, m \rangle, p) = 0,$$

then $\omega \cdot \mathbf{R}_x(\langle G, m_T \rangle, p) = 0$ too. Since the first $2|V|$ columns are exactly as in the fixed torus case, we need only show this holds for the new column.

Consider the column sum corresponding to the columns of the lattice elements in $\mathbf{R}_x(\langle G, m_T \rangle, p')$:

$$\sum_{e \in E} \omega_e \left(m_T(e) \left[(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e) \right] \right)_x. \quad (4)$$

Recall that $m_T(e) = m_T(v_i) + m(e) - m_T(v_j)$, where $m_T(v_i)$ represents the T -potential of the vertex v_i (the T -potential of a vertex v_i is the net gain on the directed path along T from the root vertex). Expanding (4), we obtain

$$\sum_{e \in E} \omega_e \left(m(e) [\cdots] + m_T(v_i) [\cdots] - m_T(v_j) [\cdots] \right)_x, \quad (5)$$

where $[\cdots] = [(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e)]$. We know that

$$\begin{aligned} & \sum_{e \in E} \omega_e \left(m(e) \left[(p_i + m_T(v_i)) - (p_j + m_T(v_j)) - m_T(e) \right] \right)_x \\ &= \sum_{e \in E} \omega_e \left(m(e) [p_i - p_j - m(e)] \right)_x \\ &= 0, \quad \text{since } \omega \cdot \mathbf{R}_x(\langle G, m \rangle, p) = 0. \end{aligned}$$

Now note that $m_T(v_i)$ and $m_T(v_j)$ have one of $|V|$ different values. Grouping (5) according to these values, we obtain

$$\sum_{i=1}^{|V|} (m_T(v_i))_x \left[\sum_{j: (i,j) \in E} \omega_e (p_i + m_T(v_i) - (p_j + m_T(v_j)) - m_T(e)) \right]_x,$$

where each edge is counted exactly twice, once for its initial vertex and once for its terminal vertex, with sign depending on the orientation of the edge. But by (3), the sum inside the square brackets is zero, since it represents the column sum at any vertex. Hence (4) is also zero. The same argument also works in reverse, which proves the claim. \square

2.4 Periodic Orbit Frameworks on the Fixed Torus \mathcal{T}_0^2

From the rigidity matrix for the x -variable torus, we can obtain the rigidity matrix for frameworks on the fixed torus, simply by striking out the column corresponding to $x(t)$. We are left with an $|E| \times 2|V|$ matrix \mathbf{R}_0 , and a periodic orbit framework $(\langle G, m \rangle, p)$ is infinitesimally rigid on \mathcal{T}_0^2 if and only if the rank of $\mathbf{R}_0(\langle G, m \rangle, p)$ is $2|V| - 2$ [20]. Frameworks on the fixed torus are the subject of the papers [20, 19], and we record only the main result.

For brevity throughout we use the following terminology. Let $G = (V, E)$ be a graph. We say that G is (k, ℓ) -sparse if all subgraphs $G' = (V', E')$ of G satisfy $|E'| = k|V'| - \ell$. If in addition, G satisfies $|E| = k|V| - \ell$, we say that G is (k, ℓ) -tight.

Let $\langle G, m \rangle$ be a periodic orbit graph, where G is $(2, 2)$ -tight. We say that the gain assignment $m : E^+ \rightarrow \mathbb{R}^2$ is *constructive* if every subgraph $G' \subset G$ with exactly $|E'| = 2|V'| - 2$ edges contains some cycle with non-trivial net gain. For example, the periodic orbit graph $\langle G, m \rangle$ pictured in Figure 2(a) has a constructive gain assignment. Note further that the T -gain procedure preserves the net gains on cycles, and therefore the graph $\langle G, m_T \rangle$ pictured in (c) also has a constructive gain assignment.

Theorem 2.3. *The periodic orbit graph $\langle G, m \rangle$ is minimally rigid on the fixed torus \mathcal{T}_0^2 if and only if G is $(2, 2)$ -tight, and m is a constructive gain assignment.*

2.5 1-Dimensional Frameworks

The basic ideas of the rigidity of finite graphs on the line can be found in [26] or [7]. The key result is that a graph G is rigid as a 1-dimensional framework if and only if it is connected. 1-dimensional periodic frameworks were discussed in [18] and [14]. We give a brief summary.

Just as we map 2-periodic frameworks onto the torus, we may view 1-periodic frameworks as graphs on a circle. Such graphs may be on a circle of fixed circumference x (*the fixed circle*), or they may be on a circle that is allowed to change circumference $x(t)$ (*the flexible circle*). We denote the fixed circle by \mathcal{T}_0^1 , and the flexible circle by \mathcal{T}^1 .

In either case, a 1-periodic orbit framework is the pair $(\langle G, m \rangle, p)$, with $m : E \rightarrow \mathbb{Z}$, and $p : V \rightarrow [0, x]$, where x is either a fixed element of \mathbb{R} for the fixed circle, or $x = x(t)$ is a continuous function of time for the flexible circle. We assume further that p maps the endpoints of any edge to distinct locations in $[0, x]$, thereby avoiding edges of length zero.

Consider a periodic orbit framework $(\langle G, m \rangle, p)$ on the fixed circle. For consistency with the notation used in the remainder of this paper, let L_0 be

the 1×1 matrix $[x]$, where the circumference of the fixed circle is x . The $|E| \times |V|$ rigidity matrix in this case will have one row corresponding to each edge $\{i, j; m\}$:

$$\begin{pmatrix} & i & & j & \\ 0 \cdots 0 & p_i - (p_j + mL_0) & 0 \cdots 0 & (p_j + mL_0) - p_i & 0 \cdots 0 \end{pmatrix},$$

where $p_i, p_j \in \mathbb{R}$, and $m \in \mathbb{Z}$. Since there is always a 1-dimensional space of trivial infinitesimal motions generated by the vector $(1, \dots, 1)^T$, the rigidity matrix has maximum rank $|V| - 1$.

Proposition 2.4. *The periodic orbit framework $(\langle G, m \rangle, p)$ is (infinitesimally) rigid on \mathcal{T}_0^1 if and only if G is connected.*

If we allow the radius of the circle to change size, in addition to connectivity, we now require the graph to “wrap” in a non-trivial fashion around the circle. That is, $\langle G, m \rangle$ must contain a constructive cycle. The rigidity matrix now has an extra column corresponding to $x(t)$, with entry (for the edge $\{i, j; m\}$) given by $m(p_i - (p_j + mL))$.

One way to see the necessity of a constructive cycle is to perform the T -gain procedure on the edges of a periodic orbit graph $\langle G, m \rangle$ with $|E| = |V|$. If no cycle is constructive, the column corresponding to $x(t)$ will be identically zero.

Proposition 2.5. *The periodic orbit framework $(\langle G, m \rangle, p)$ is (infinitesimally) rigid on \mathcal{T}^1 if and only if G is connected, and G contains a constructive cycle.*

3 Necessary Conditions for Rigidity on \mathcal{T}_x^2

Let $\langle G, m \rangle$ be a periodic orbit framework where G is $P(2, 1)$. We say m is \mathcal{T}_x^2 -constructive if

- i) every $(2, 2)$ -subgraph is constructive (i.e. every $(2, 2)$ -subgraph contains some cycle with non-trivial net gain) and
- ii) every $(2, 1)$ -subgraph is x -constructive (i.e. contains some cycle with non-trivial net gain in the x -direction.)

In [13] it was shown that the class of (k, ℓ) -tight graphs forms a matroid for natural numbers $0 \leq \ell < 2k$. Thus we say that a $(k, \ell - 1)$ -circuit is a (k, ℓ) -tight graph in which deleting any edge gives a $(k, \ell - 1)$ -tight graph. Crucial to us will be the following weaker definition.

We will say that G is a $P(2, 1)$ -graph if G is $(2, 1)$ -tight and there exists $e \in E$ such that $G - e$ is $(2, 2)$ -tight. The following proposition provides necessary conditions for minimal rigidity on \mathcal{T}_x^2 .

Proposition 3.1. *Let $(\langle G, m \rangle, p)$ be a periodic orbit framework. If $(\langle G, m \rangle, p)$ is minimally rigid on \mathcal{T}_x^2 , then G is $P(2, 1)$ and m is \mathcal{T}_x^2 -constructive.*

Proof. That G must be $P(2, 1)$ for $(\langle G, m \rangle, p)$ to be rigid follows from Theorem 3.3 and Lemma 3.4. Let H be the unique $(2, 2)$ -circuit.

To see that $\langle G, m \rangle$ is \mathcal{T}_x^2 -constructive, we show that H is x -constructive, and that every $(2, 2)$ -tight subgraph of G obtained by deleting a single edge of H is constructive.

To see that H is x -constructive, suppose toward a contradiction that $\langle G, m \rangle$ contains no x -constructive cycle. Applying the T -gain procedure to any tree in G will produce a T -gain assignment m_T , where all x -coordinates are zero. Then all of the entries of the single lattice column of the rigidity matrix will be zero. Hence we effectively have $2|V| - 1$ edges in the fixed torus rigidity matrix, which has maximum rank $2|V| - 2$, a contradiction.

Since G is $P(2, 1)$, deleting any edge e from H results in a $(2, 2)$ -subgraph of G . Since every such subgraph must correspond to a set of linearly independent rows in the rigidity matrix, Proposition 3.2 implies that the gain assignment m restricted to this subgraph must be constructive. \square

When we move from the fixed torus to the variable torus, we add columns to the rigidity matrix. As a result, it seems possible that edges that were dependent on the fixed torus become independent on the variable torus. When $d = 2$, and our dependent subgraphs are of size $2|V| - 2$, this is not the case.

Proposition 3.2. *Let $\langle G, m \rangle$ be a $(2, 2)$ -tight periodic orbit graph. If $\langle G, m \rangle$ is dependent on \mathcal{T}_0^2 then $\langle G, m \rangle$ is also dependent on \mathcal{T}_x^2 .*

Proof. Suppose that $\langle G, m \rangle$ is dependent on \mathcal{T}_0^2 . Then there is some subgraph $\langle G', m' \rangle \subseteq \langle G, m \rangle$ with $|E'| = 2|V'| - 2$, and no constructive cycle. Therefore, all gains on this subgraph are T -gain equivalent to $(0, 0)$. Then the entries in the lattice column of the rigidity matrix corresponding to these edges will be zero, since $(m_T)_x = 0$ for all e , and therefore the edges continue to be dependent on \mathcal{T}_x^2 . \square

A *map-graph* is a graph in which each connected component has exactly one cycle. By a result of Whiteley [25], $G = (V, E)$ is a (k, ℓ) -tight graph if and only if E is the edge-disjoint union of ℓ spanning trees and $k - \ell$ spanning map-graphs. Note that each map-graph need *not* be connected. This is in contrast to the situation for minimally rigid periodic orbit frameworks on \mathcal{T}_x^2 :

Theorem 3.3. *Let $(\langle G, m \rangle, p)$ be a minimally rigid framework on the variable torus \mathcal{T}_x^2 . Then the edges of $\langle G, m \rangle$ admit a decomposition into one spanning tree and one connected spanning map-graph.*

Proof. This proof is similar to the proof of Theorem 2.18 in [24]. Let $(\langle G, m \rangle, p)$ be a minimally rigid framework on \mathcal{T}_x^2 . The rigidity matrix, $\mathbf{R}_x(\langle G, m \rangle, p)$ has rank $2|V| - 1$, and dimension $(2|V| - 1) \times (2|V| + 1)$, with $2|V|$ columns corresponding to the vertices, and one column corresponding to the flexibility of the lattice. Adding the two rows

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

has the effect of eliminating the 2-dimensional space of infinitesimal translations. This “tie down” is described in [24], and is equivalent to pinning one vertex on the torus. The resulting square matrix has $2|V|+1$ independent rows, and hence a non-zero determinant.

Reorder the columns of $\mathbf{R}_x(\langle G, m \rangle, p)$ by coordinates, with the single lattice column grouped with the first coordinates. Regard the determinant as a Laplace decomposition where the terms are products of the determinants of one square block M_i with dimension $(|V|+1) \times (|V|+1)$, and one square block N_i with dimension $|V| \times |V|$. That is, $\det \mathbf{R}_x(\langle G, m \rangle, p) = \sum M_i N_i$. The block M_i contains all of the entries from the columns of the first coordinates, and the N_i block contains the second coordinates. Each block contains a single tie-down row.

There must be at least one nonzero product $M_k N_k$. By the Laplace decomposition, the rows used in M_k and N_k form disjoint subgraphs, and M_k or N_k will each contain one of the tie-down rows. The $|V|$ rows of M_k that are not tie-down rows have rank $|V|$, and this submatrix corresponds to the rigidity matrix of a 1-dimensional graph on the flexible circle (the periodic line). By Proposition 2.5, we know that such a graph must be connected, and must contain a constructive cycle. Hence the $|V|$ edges of the M_k block form a spanning connected map-graph.

Similarly, the $|V|-1$ edges of the N_k block that are not tie-down edges correspond to the rigidity matrix of a graph on the fixed circle. Since the block N_k has rank $|V|$, the edges that are not tie-down edges are independent on the fixed circle, and hence by Proposition 2.4 the graph is connected. Therefore the edges of the N_k block form a spanning tree of G . \square

Lemma 3.4. *Suppose G has $|E| = 2|V| - 1$. If the edges of G admit a decomposition into one (edge-disjoint) spanning tree and one connected spanning map-graph, then G is a $P(2, 1)$ -graph and G contains a unique $(2, 2)$ -circuit.*

Proof. In light of Theorem 5.11, the lemma is simply a re-statement of Lemma 5.1. \square

4 H1, H2 Preserve Rigidity on \mathcal{T}_x^2

We will use $d(v)$ to denote the degree of the vertex v and $d_G(v)$ when the context of the graph is not clear. $N(v)$ denotes the set of neighbours of v . As is common in the literature we will refer to the following construction moves as *Henneberg operations*:

- (1a) add a vertex v_0 with $d(v_0) = 2$ and $N(v_0) = \{v_1, v_2\}$, $v_1 \neq v_2$,
- (1b) add a vertex v_0 with $d(v_0) = 2$ and $N(v_0) = \{v_1\}$,
- (2a) remove an edge $v_1 v_2$, $v_1 \neq v_2$, and add a vertex v_0 with $d(v_0) = 3$ and $N(v_0) = \{v_1, v_2, v_3\}$ for some $v_3 \in V$,

- (2b) remove an edge v_1v_2 , $v_1 \neq v_2$, and add a vertex v_0 with $d(v_0) = 3$ and $N(v_0) = \{v_1, v_2\}$ with one edge connecting v_0 with v_1 , and two edges connecting vertices v_0 and v_2 .

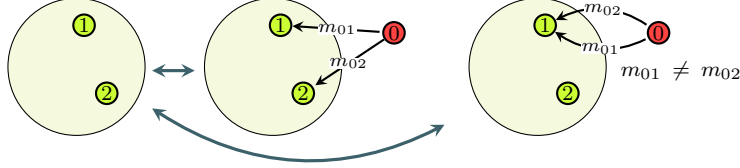


Figure 3: The $H1$ moves (periodic vertex addition). The large circular region represents a generically rigid periodic orbit graph on \mathcal{T}_x^2 .

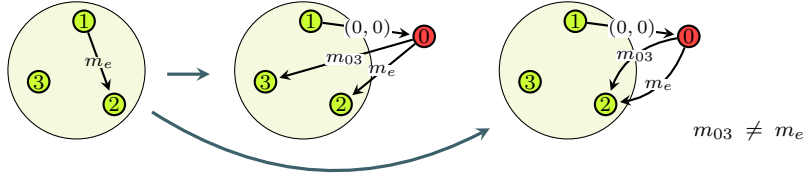


Figure 4: The $H2$ moves (periodic edge split). The gain m_e on the edge connecting 1 and 2 is preserved through this split.

More strongly, when the moves are applied to a periodic orbit framework $\langle G, m \rangle$ with changes to the gains as illustrated in Figures 3 and 4, we say that these operations are *gain-preserving Henneberg operations*.

It was shown in [20] using linear algebra techniques that gain-preserving Henneberg operations preserve the maximality of the rank of the rigidity matrix, echoing the situation for finite frameworks. The corresponding results for the variable torus \mathcal{T}_x^2 can be proven entirely similarly; we leave the details to the reader.

Proposition 4.1. *Let $\langle G, m \rangle$ be a periodic orbit and let $\langle G', m' \rangle$ be the result of a gain-preserving $H1$ operation on $\langle G, m \rangle$. Let p be generic and let $p' = (p, p_{n+1})$ be chosen generically with respect to p . Then the rows of $\mathbf{R}_x(\langle G, m \rangle, p)$ are linearly independent if and only if the rows of $\mathbf{R}_x(\langle G', m' \rangle, p')$ are linearly independent.*

Proposition 4.2. *Let $\langle G, m \rangle$ be a periodic orbit and let $\langle G', m' \rangle$ be the result of a gain-preserving $H2$ operation on $\langle G, m \rangle$. Let p be generic and let $p' = (p, p_{n+1})$ be chosen generically with respect to p . If the rows of $\mathbf{R}_x(\langle G, m \rangle, p)$ are linearly independent then the rows of $\mathbf{R}_x(\langle G', m' \rangle, p')$ are linearly independent.*

5 $P(2, 1)$ Graphs

In this section we consider in detail the structure of $P(2, 1)$ -graphs. For $X \subset V$, $G[X]$ denotes the subgraph induced by X . For two subsets $J, K \subset V$, $d(J, K)$ denotes the number of edges in E with one end-vertex in $G[J]$ and one in $G[K]$. For a subset $X \subset V$ let $i(X)$ denote the number of edges in the subgraph induced by X . Observe first that a $P(2, 1)$ -graph can have at most one loop and a $(2, 2)$ -circuit can have a loop if and only if the graph is a single loop.

5.1 Critical Sets

Let $G = (V, E)$ be a $P(2, 1)$ -graph and let $X \subset V$. We will say X is *over-critical* if $i(X) = 2|X| - 1$, *critical* if $i(X) = 2|X| - 2$ and *semi-critical* if $i(X) = 2|X| - 3$. In each case if there is a degree 3 vertex $v \in V$ with $N(v) = \{x, y, z\}$ and X contains x, y but not z, v then we say that X is *over- v -critical*, *v -critical* or *semi- v -critical* respectively.

To simplify the arguments in Section 6 we record some basic facts about $P(2, 1)$ -graphs. In the next lemma, minimal means having the least number of vertices.

Lemma 5.1. *Let $G = (V, E)$ be a $P(2, 1)$ -graph. Then there is a unique minimal over-critical set $X \subset V$ and $G[X]$ is a $(2, 2)$ -circuit.*

Proof. If G is a $(2, 2)$ -circuit, then $X = V$. Otherwise by definition there exists $X \subsetneq V$ with $i(X) = 2|X| - 1$. Choose the minimal over-critical set $X^- \subset X$. This induces a $(2, 2)$ -circuit.

For uniqueness suppose $J \subsetneq V$ such that $J \neq X^-$ is over-critical. If J and X^- are disjoint we contradict the definition of a $P(2, 1)$ -graph. Thus $i(X^- \cup J) + i(X^- \cap J) = i(X^-) + i(J) + d(X^- - J, J - X^-) = 2|X^-| - 1 + 2|J| - 1 + d(X^- - J, J - X^-)$. This implies that $d(X^- - J, J - X^-) = 0$, $i(X^- \cup J) = 2|X^- \cup J| - 1$ and $i(X^- \cap J) = 2|X^- \cap J| - 1$ contradicting the minimality of X^- . \square

Lemma 5.2. *Let $G = (V, E)$ be a $P(2, 1)$ -graph. Then*

- (1) G is 2-edge-connected.
- (2) If $X \subset V$ is critical then $G[X]$ is connected.
- (3) If $X \subset V$ is semi-critical then either $G[X]$ is connected or X has two connected components A, B such that A is over-critical and B is critical.

Proof. In each case Lemma 5.1 will imply at most one of A, B is over-critical.

For (1) suppose $V = A \cup B$ for $A, B \subset V$ with $A \cap B = \emptyset$ and $d(A, B) = 1$. Now

$$\begin{aligned} 2|V| - 1 = |E| &= i(A) + i(B) + 1 \\ &\leq 2|A| - 1 + 2|B| - 2 + 1 \\ &= 2|V| - 2, \end{aligned}$$

a contradiction.

For (2) suppose $V = A \cup B$ for $A, B \subset V$ with $A \cap B = \emptyset$ and $d(A, B) = 0$.
Now

$$\begin{aligned} 2|X| - 2 = i(X) &= i(A) + i(B) \\ &\leq 2|A| - 1 + 2|B| - 2 \\ &= 2|X| - 3, \end{aligned}$$

a contradiction.

For (3) suppose $G[X]$ is not connected. Then the proof is entirely similar to (2). \square

However $P(2, 1)$ -graphs need not be 2-connected and need not be 3-edge-connected.

Lemma 5.3. *Let $G = (V, E)$ be a $P(2, 1)$ -graph with unique $(2, 2)$ -circuit $G[X]$ and let $v \in V$ have $N(v) = \{x, y, z\}$. Let $x, y, z \in X$. Then*

- (1) $v \in X$ and there is no over- v -critical set.
- (2) If Y is v -critical on x, y but not z with no over-critical subset and Z is semi- v -critical on x, z but not y with no critical subset then
 - (a) if $|Y \cap Z| > 1$ then $Y \cup Z$ is critical and $Y \cap Z$ is semi-critical.
 - (b) if $|Y \cap Z| = 1$ then $Y \cup Z$ is semi-critical and $Y \cap Z$ is critical.

Proof. First $v \in X$ otherwise $i(X \cup v) = 2|X \cup v|$ and any over- v -critical set contradicts Lemma 5.1.

For (2) $Y \cap Z \subset Z$ so $i(Y \cap Z) \leq 2|Y \cap Z| - 3$. Since

$$i(Y \cup Z) + i(Y \cap Z) = 2|Y \cup Z| + 2|Y \cap Z| - 5 + d(Y, Z)$$

and $Y \cup Z$ contains x, y, z but not v we deduce that $Y \cup Z$ is critical and $Y \cap Z$ is semi-critical. (b) is entirely similar. \square

Lemma 5.4. *Let $G = (V, E)$ be a $P(2, 1)$ -graph with unique $(2, 2)$ -circuit $G[X]$ and let $v \in V$ have $N(v) = \{x, y, z\}$. Let $x, y \in X, z \in V - X$. Then*

- (1) $v \in V - X$ and X is the unique over- v -critical set,
- (2) there is no v -critical set containing z ,
- (3) if $Y \subset V$ contains x, z but not y, v and $|X \cap Y| > 1$ then $G[Y]$ is not $(2, 3)$ -tight.

Proof. First $v \in V - X$ contradicts Lemma 5.1 and X is unique by Lemma 5.1 and the definition of a $(2, 2)$ -circuit.

For (2) suppose there is such a v -critical set Z .

$$\begin{aligned} i(X \cup Z) + i(X \cap Z) &= i(X) + i(Z) + d(X - Z, Z - X) \\ &\leq 2|X| - 1 + 2|Z| - 2 + d(X - Z, Z - X). \end{aligned}$$

Since $i(X \cap Z) \leq 2|X \cap Z| - 2$ we have $d(X - Z, Z - X) = 0$ and $i(X \cup Z) = 2|X \cup Z| - 1$ but $i(X \cup Z \cup v) = 2|X \cup Z \cup v|$, a contradiction.

Finally, since $|X \cap Y| > 1$ we have $i(X \cap Y) \leq 2|X \cap Y| - 3$ as $X \cap Y \subset Y$. Thus, similarly to before, $X \cup Y$ is over v -critical, and adding back v gives a contradiction. \square

Lemma 5.5. *Let $G = (V, E)$ be a $P(2, 1)$ -graph with unique $(2, 2)$ -circuit $G[X]$ and let $v \in V$ have $N(v) = \{x, y, z\}$. Let $y, z \in V - X$. Then $v \in V - X$, there is at most one over- v -critical set Y . Moreover if there is such a Y then there is no v -critical set and at most one semi- v -critical set.*

Proof. First $v \in V - X$ otherwise $i(X - v) > 2|X - v| - 1$. Let Y be over- v -critical containing x, z but not y . If W is over- v -critical on x, y but not z then $i(Y \cup W \cup v) \geq 2|Y \cup W \cup v|$.

Now suppose Z is v -critical on x, y but not z . Then $i(Y \cap Z) \leq 2|Y \cap Z| - 2$ so $i(Y \cup Z) = 2|Y \cup Z| - 1$ and adding back v creates a contradiction.

Finally suppose Z is semi- v -critical on x, y but not z . Then $Y \cup Z$ is critical. Consider $W \subset V$ containing y, z but not x, v . $i(Y \cap W) \leq 2|Y \cap W| - 2$ and $i(Z \cap W) \leq 2|Z \cap W| - 2$. Therefore $i(Y \cup Z \cup W) \leq 2|Y \cup Z \cup W| - 4$. \square

Lemma 5.6. *Let $G = (V, E)$ be a $P(2, 1)$ -graph containing a unique $(2, 2)$ -circuit $G[X]$, with a degree 3 vertex v with $N(v) = \{x, y, z\}$ such that $x, y \in X, z \notin X$. Let $Y_{xz} \subset V$ contain x, z but not y, v and $Y_{yz} \subset V$ contain y, z but not x, v . Then at most one of Y_{xz} and Y_{yz} is semi-critical.*

Proof. Suppose Y_{yz} is semi-critical. Firstly if $i(Y_{xz}) < 2|Y_{xz}| - 3$ then $i(X \cap Y_{xz}) \geq 2|X \cap Y_{xz}| - 2$ by Lemma 5.1. Thus $i(X \cup Y_{xz}) < 2|X \cup Y_{xz}| - 2$ so adding back v and its 3 edges contradicts the definition of a $P(2, 1)$ -graph. Suppose $i(Y_{xz}) = 2|Y_{xz}| - 3$.

$$\begin{aligned} i(X \cup Y_{xz}) + i(X \cap Y_{xz}) &= i(X) + i(Y_{xz}) + d(X - Y_{xz}, Y_{xz} - X) \\ &= 4 + d(X - Y_{xz}, Y_{xz} - X). \end{aligned}$$

If $i(X \cup Y_{xz}) = 2|X \cup Y_{xz}| - 1$ add v and its 3 edges for a contradiction. If $i(X \cap Y_{xz}) = 2|X \cap Y_{xz}| - 1$ then we contradict $G[X]$ being a $(2, 2)$ -circuit. Thus $i(X \cup Y_{xz}) = 2|X \cup Y_{xz}| - 2$ and $i(X \cap Y_{xz}) = 2|X \cap Y_{xz}| - 2$. Since $X \cap Y_{xz} \subset Y_{xz}$ we know $|X \cap Y_{xz}| = 1$. Similarly we derive that $|X \cap Y_{yz}| = 1$. Now

$$i(Y_{xz} \cup Y_{yz}) + i(Y_{xz} \cap Y_{yz}) = 6 + d(Y_{xz} - Y_{yz}, Y_{yz} - Y_{xz}).$$

$i(Y_{xz} \cap Y_{yz}) > 2|Y_{xz} \cap Y_{yz}| - 1$ by Lemma 5.1 and $i(Y_{xz} \cup Y_{yz}) > 2|Y_{xz} \cup Y_{yz}| - 1$ otherwise adding v and its 3 edges gives a contradiction. Hence

$$i(Y_{xz} \cup Y_{yz}) = 2|Y_{xz} \cup Y_{yz}| - s$$

where $s \in \{2, 3, 4\}$.

$$i(X \cup (Y_{xz} \cup Y_{yz})) = 1 + s - 4 + d(X - (Y_{xz} \cup Y_{yz}), (Y_{xz} \cup Y_{yz}) - X) \leq 1.$$

Thus adding v and its 3 edges violates the definition of a $P(2, 1)$ -graph. \square

5.2 Henneberg Operations on $P(2, 1)$ -graphs

A vertex v in a $P(2, 1)$ -graph is *admissible* if there is some inverse Henneberg operation removing v that results in a $P(2, 1)$ -graph.

For brevity we will use K_i^j to denote the complete graph on i vertices with j copies of each edge. Note when $i = 1$ then j denotes the number of loops on that single vertex.

The next lemma is useful in extending the standard arguments showing the inverse moves preserve $(2, l)$ -tightness to show that the inverse moves preserve the $P(2, 1)$ condition.

Lemma 5.7. *Let $G = (V, E)$ be a $P(2, 1)$ -graph not equal to K_2^3 with no degree 2 vertex. Then there exists a degree 3 vertex v and an edge e not incident to v such that $G - e$ is $(2, 2)$ -tight.*

Proof. This is trivial if G is a $(2, 2)$ -circuit. Suppose G is not a $(2, 2)$ -circuit then there exists an over-critical set $K \subsetneq V$. Let $J = V - K$. Now $i(V) = i(J) + i(K) + d(J, K)$ so $i(J) = d(J, K)$. Also $i(J) = d(J, K) \geq 2$ by Lemma 5.2 (1).

Suppose $d_G(v) \geq 4$ for all $v \in J$. There are at least $4|J| + 2$ vertex/edge incidences which implies $i(J) + d(J - K, K - J) \geq 2|J| + 1$ but

$$2|V| - 1 = i(V) = i(K) + i(J) + d(J - K, K - J) \geq 2|K| + 2|J|$$

so there is a vertex $v_3 \in J$ with $d_G(v_3) = 3$. Now from the definition of a $P(2, 1)$ -graph there is some edge in $G[K]$ that gives the result. \square

The following lemma allows us to apply the inverse moves easily in the case that the $P(2, 1)$ -graph G happens to be a $(2, 2)$ -circuit.

Lemma 5.8. *Let $G = (V, E)$ be a graph. If G is a $(2, 2)$ -circuit then G contains a degree 3 vertex v , and for any such v , $G - v$ is $(2, 2)$ -tight.*

Note that the lemma is not in general true for $P(2, 1)$ -graphs and that the converse fails for $(2, 2)$ -circuits but is true by definition for $P(2, 1)$ -graphs.

Proof. Let G be a $(2, 2)$ -circuit. Since $|E| = 2|V| - 1$ we have $\sum_{i=1}^{|V|} (4 - d(i)) = 2$. Therefore there exists $v \in V$ with $d(v) \leq 3$. $(2, 2)$ -circuits cannot contain vertices of degree ≤ 2 ; suppose u was such a vertex then $i(V - u) = 2|V - u| - 1$ edges contradicting the definition of a $(2, 2)$ -circuit. Thus G contains a degree 3 vertex v . Clearly $V \setminus v$ is critical since we can think of the operation of deleting a degree 3 vertex as the composition of an edge deletion and an inverse 1a move. \square

Lemma 5.9. *Let G be a $(2, 2)$ -circuit. Then either $G = K_2^3$ or there is an inverse 2a or 2b move on any degree 3 vertex that results in a $P(2, 1)$ -graph.*

Proof. Let $v \in V$ have $d(v) = 3$. $V - v$ is critical by Lemma 5.8. If $N(v) = \{a\}$ then $v \in K_2^3$. If not then adding any non-edge between the neighbours of v creates a $P(2, 1)$ -graph. \square

Note the stronger statement that every degree 3 vertex in a $(2, 2)$ -circuit is admissible is false, for similar considerations see [2] and [16].

In the following lemma, recall it is well known that if G is $(2, \ell)$ -tight ($\ell = 2, 1$) then there is an inverse 2a or 2b move that results in a $(2, \ell)$ -tight graph. Hence we need only concern ourselves with the additional subgraph condition.

Lemma 5.10. *Let $G = (V, E)$ be a $P(2, 1)$ -graph containing a vertex v with $d(v) = 3$ and $|N(v)| > 1$. Then v is admissible.*

Proof. By Lemma 5.9 we may assume G is not a $(2, 2)$ -circuit. Thus by Lemma 5.1 G contains a unique $(2, 2)$ -circuit $G[X]$. By Lemma 5.7 we find a vertex v in $V - X$ with $d(v) = 3$. For any edge e in $G[X]$, $G - e$ is $(2, 2)$ -tight and $G - e - v + xy$ for some $x, y \in N(v)$ is $(2, 2)$ -tight. Therefore $G - v + xy$ is a $P(2, 1)$ -graph. □

For brevity we use the terminology *leaf* for a degree 1 vertex. We can now collect together the results in this section into our main result about $P(2, 1)$ -graphs. $(1) \Leftrightarrow (3)$ is a slight refinement of a result of Whiteley.

Theorem 5.11. *Let $G = (V, E)$. The following are equivalent:*

1. G is a $P(2, 1)$ -graph,
2. G can be constructed from K_1^1 or K_2^3 by 1a, 1b, 2a and 2b moves,
3. G is the edge disjoint union of a spanning tree T and a connected map graph M .

Proof. First we prove $(1) \Leftrightarrow (2)$. It is easy to see that any of these four moves applied to an arbitrary $P(2, 1)$ -graph results in a $P(2, 1)$ graph. Since K_1^1 and K_2^3 are $P(2, 1)$ -graphs it follows that any graph constructed from a sequence of these moves is a $P(2, 1)$ -graph.

The converse follows from the above sequence of results by induction on $|V|$. Suppose G is a $P(2, 1)$ -graph containing a loop. Then this loop is the unique $(2, 2)$ -circuit within G and Lemma 5.10 guarantees an inverse move. Suppose now that G is loopless. Then either G is a $(2, 2)$ -circuit in which case apply Lemma 5.9 or G contains a unique $(2, 2)$ -circuit and apply Lemma 5.7. The result follows from Lemma 5.10.

$(3) \Rightarrow (1)$ follows since $|E| = |E(T)| + |E(M)|$, $|E(T)| = |V| - 1$, $|E(T')| \leq |V(T')| - 1$ for any subgraph T' of T , $|E(M)| = |V|$ and $|E(M')| \leq |V(M')|$ for any subgraph M' of M .

Let G' be formed from G by one of the four construction moves. $(2) \Rightarrow (3)$ follows by showing that in each case if G satisfies (3) then so does G' . This is trivial in each case. Particularly in the 1a and 1b moves the new vertex is a leaf in the tree and in the connected map graph. In the 2a and 2b moves removing xy the new edges xv, yv go in whichever of T or M contained xy and the remaining edge goes in the other. □

6 G is $P(2, 1)$ and \mathcal{T}_x^2 -constructive \iff H1, H2

Let $\langle G, m \rangle$ be a \mathcal{T}_x^2 orbit graph, where G is $P(2, 1)$ and m is \mathcal{T}_x^2 -constructive. We say that a vertex $v \in V$ is a *circuit vertex* if it is contained within the minimal $(2, 1)$ -tight subgraph of G (this subgraph is a $(2, 2)$ -circuit). Recall the Henneberg operations defined in Section 4. As we will see there is an infinite but controllable class of periodic orbit frameworks which are $P(2, 1)$ -graphs and \mathcal{T}_x -constructive, yet for which the Henneberg operations are insufficient, see Figure 6. For these graphs we must introduce an additional $H2$ Henneberg type move, see Figure 7 for the definition. In the following theorem, our main result, by Henneberg operation we mean $H1$ or $H2$ move.

Theorem 6.1 (\mathcal{T}_x^2 Henneberg Theorem). *Let $\langle G, m \rangle$ be a periodic orbit framework. G is a $P(2, 1)$ -graph and m is \mathcal{T}_x -constructive if and only if G can be generated from a single loop by gain-preserving Henneberg operations.*

Given Theorem 6.1 we may now prove Theorem 1.2.

Proof of Theorem 1.2. The necessity of the construction operations follows from Proposition 3.1 and the fact that each of the construction operations takes a \mathcal{T}_x^2 -constructive $P(2, 1)$ -graph to a \mathcal{T}_x^2 -constructive $P(2, 1)$ -graph.

By Theorem 2.1, Propositions 4.1 and 4.2 show that periodic rigidity is preserved by the Henneberg operations. Thus the sufficiency follows from Theorem 6.1. \square

The remainder of this section will prove Theorem 6.1.

6.1 Paths, Cycles and Gains

Lemma 6.2. *Let $\langle G, m \rangle$ be a periodic orbit framework where $G = (V, E)$ is a $P(2, 1)$ graph with unique minimal over-critical set $X \subset V$ and m is \mathcal{T}_x -constructive. Let $v_0 \in V$ have $N(v_0) = \{v_1, v_2, v_3\} \subset X$. Let G_1^1 be $(2, 2)$ -tight containing v_1, v_2 but not v_0, v_3 with no x -constructive cycle and all paths from v_1 to v_2 have x -gain $(m_2 - m_1)_x$. Similarly let G_2^1 be $(2, 2)$ -tight containing v_2, v_3 but not v_0, v_1 with no x -constructive cycle and all paths from v_1 to v_2 have x -gain $(m_3 - m_2)_x$. Then all paths from v_1 to v_3 have x -gain $(m_3 - m_1)_x$.*

Proof. Let P be a path from v_1 to v_3 . The path must pass through $G_1^1 \cap G_2^1$, say through the vertex $u \in V_1^1 \cap V_2^1$. See Figure 5. In the simplest case, the path $v_1 \rightarrow u$ lies completely within G_1^1 , and the path $u \rightarrow v_3$ lies completely within G_2^1 . Since the graph $G_1^1 \cap G_2^1$ is connected, there exists a path $P' \in G_1^1 \cap G_2^1$ from $u \rightarrow v_2$. Suppose the gains on the path are as follows:

$$\begin{aligned} v_1 &\xrightarrow{m_a} u \xrightarrow{m_b} v_3 \\ v_1 &\xrightarrow{m_a} u \xrightarrow{m_c} v_2 \xrightarrow{-m_c} u \xrightarrow{m_b} v_3. \end{aligned}$$

But it is now clear that

$$(m_a + m_c)_x = (m_2 - m_1)_x, \text{ and } (m_b - m_c)_x = (m_3 - m_2)_x.$$

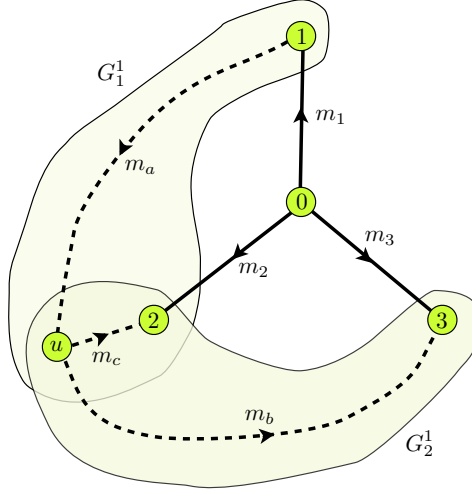


Figure 5: Proof of Lemma 6.2. Both G_1^1 and G_2^1 are (2,2)-tight.

Summing the two equations, we find that $m_a + m_b = m_3 - m_1$, as desired.

Now consider the case that the path P is as follows:

$$v_1 \rightarrow u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_n \rightarrow v_3,$$

where $v_1 \rightarrow u_1$ is contained within G_1^1 , the path $u_1 \rightarrow u_2 \in G_2^1$, and the remaining segments alternate between G_1^1 and G_2^1 , with the final path $u_n \rightarrow v_3 \in G_2^1$.

Note first that $v_2 \rightarrow u_1 \rightarrow u_2 \rightarrow v_2$ is completely contained within G_2^1 , and therefore it must have trivial net x -gain. Similarly, the net x -gain on any closed path

$$\underbrace{v_2 \rightarrow u_i \rightarrow u_{i+1} \rightarrow v_2}_0 \quad (6)$$

is trivial. Then the net x -gain on the path from v_1 to v_3 is as follows:

$$\underbrace{v_1 \rightarrow u_1 \rightarrow v_2}_{m_2 - m_1} \rightarrow u_1 \rightarrow u_2 \rightarrow v_2 \rightarrow \cdots v_2 \rightarrow u_{n-2} \rightarrow u_{n-1} \rightarrow \underbrace{v_2 \rightarrow u_n \rightarrow v_3}_{m_3 - m_2},$$

and all the paths in the middle contribute nothing to the net x -gain by (6), which completes the proof. \square

Note that the only graph theoretical consideration in the proof was the fact that $G_1^1 \cap G_2^1$ is connected. Thus, in view of the results in Subsection 5.1, the argument adapts easily to each of the other cases we will consider.

6.2 Two Distinct Neighbours

In approaching this case, we need to pay special attention to a particular class of graphs which we call the *bunny ears* class. It consists of a three-valent vertex v_0 , which is adjacent only to the vertices v_1 , and v_2 . The vertices v_1 and v_2 are both members of two edge-disjoint $(2,3)$ -tight subgraphs, which comprise the rest of the graph (see Figure 6(a)). The simplest example, which gives the class its name, consists of only five edges, as shown in Figure 6(b).

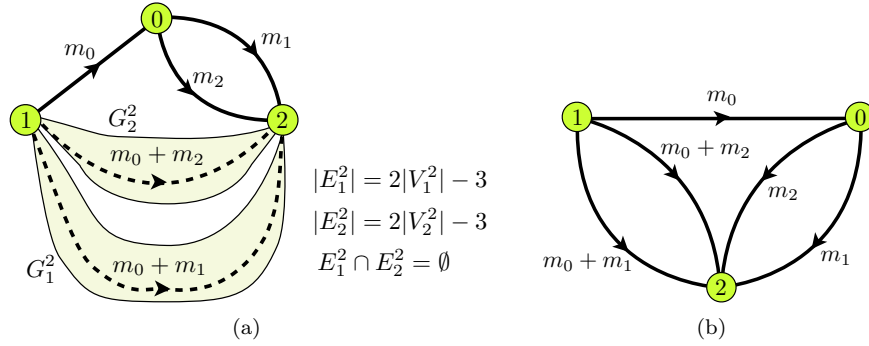


Figure 6: The bunny ears class of graphs (a). The smallest example, where G_1^2 and G_2^2 consist of single edges is shown in (b). Note that attempting to delete either vertex 0 or 1 in a reverse $H2$ move will produce a dependent $(2,1)$ -tight subgraph. We define a Henneberg 2c move to deal with this case, see Figure 7.

For this class of graphs only, we introduce one additional Henneberg move (*Henneberg 2c*), which is essentially an edge split on a loop edge. We will use the reverse move: deleting a three-valent vertex and adding a loop edge. See Figure 7.

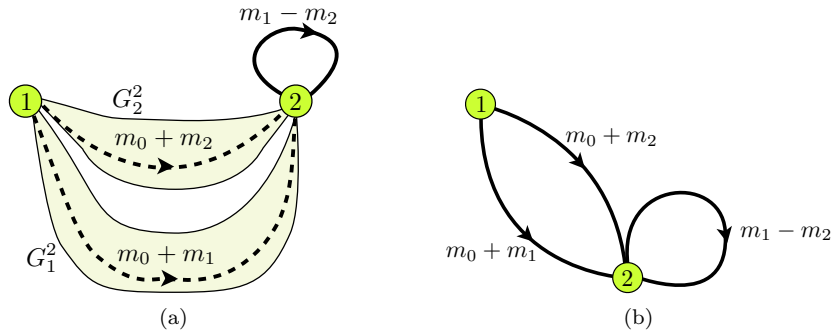


Figure 7: The reverse Henneberg 2c move on the bunny ears class of graphs (a). The reverse Henneberg 2c move on the smallest example (b).

Proposition 6.3. *Let $\langle G, m \rangle$ be a periodic orbit framework where $G = (V, E)$ is a $P(2, 1)$ graph, and m is \mathcal{T}_x -constructive. Let v_0 be a three-valent vertex adjacent to two distinct vertices v_1 and v_2 , and suppose that v_0, v_1, v_2 are circuit vertices. Let the edges adjacent to v_0 be*

$$\{v_0, v_1; m_0\}, \{v_0, v_2; m_1\}, \{v_0, v_2; m_2\}.$$

Then v_0 is admissible unless v_0, v_1, v_2 are part of the bunny ears class of graphs, in which case we add the (loop) edge

$$\{v_2, v_2; m_1 - m_2\}.$$

Proof. The candidate edges for a reverse $H2$ move deleting v_0 are

$$\{v_1, v_2; m_0 + m_1\}, \{v_1, v_2; m_0 + m_2\}.$$

Toward a contradiction, suppose we cannot add either edge. Then there are subgraphs of G which prevent the addition of these edges. Note first that there is no $(2, 1)$ -tight subgraph of G containing v_1 and v_2 but *not* the vertex v_0 , since this would contradict the fact that G is $(2, 1)$ -tight.

If we cannot add the edge $\{v_1, v_2; m_0 + m_1\}$, then we must have one of two scenarios:

- There is a subgraph $G_1^1 \subset G$ which contains v_1, v_2 but not v_0 , and satisfying $i(V_1^1) = 2|V_1^1| - 2$ that will not have an x -constructive gain assignment with the addition of the candidate edge $\{v_1, v_2; m_0 + m_1\}$. That is, G_1^1 is $(2, 2)$ -tight, contains no x -constructive cycles, and all paths from v_1 to v_2 have x -gain $(m_0 + m_1)_x$. Note that G_1^1 cannot contain a $(2, 1)$ subgraph, since that subgraph would be x -constructive by hypothesis.
- There is a $(2, 3)$ -tight subgraph $G_1^2 \subset G$ which contains v_1, v_2 but not v_0 , that will not have a constructive gain assignment with the addition of $\{v_1, v_2; m_0 + m_1\}$. That is, G_1^2 contains no constructive cycles, and all paths from v_1 to v_2 have net gain $m_0 + m_1$.

Similarly, if we cannot add the edge $\{v_1, v_2; m_0 + m_2\}$, then we have the two scenarios:

- G_2^1 is a $(2, 2)$ -tight subgraph containing v_1, v_2 but not v_0 , and with all paths from v_1 to v_2 having x -gain $(m_0 + m_2)_x$.
- G_2^2 is a $(2, 3)$ -tight subgraph containing v_1, v_2 but not v_0 , and with all paths from v_1 to v_2 having net gain $m_0 + m_2$.

The subscripts of the subgraphs correspond to which edge we are trying to put in, and the superscripts tell us how many edges we can add before creating an overbraced framework. If we cannot put in either edge, then we have three cases, corresponding to the possible pairs of the four subgraphs above:

- (1) G_1^1 and G_2^1
- (2) G_1^1 and G_2^2 (and by symmetry G_1^2 and G_2^1)
- (3) G_1^2 and G_2^2

Case (1) $V_1^1 \cap V_2^1$ and $V_1^1 \cup V_2^1$ are both critical, otherwise adding back the vertex v_0 and its three adjacent edges provides a contradiction. Further $G_1^1 \cap G_2^1$ is connected by Lemma 5.2 (2). Therefore, any path from v_1 to v_2 in $G_1^1 \cap G_2^1$ is also a path in both G_1^1 and G_2^1 . Hence all paths from v_1 to v_2 have x -gain $(m_0 + m_1)_x = (m_0 + m_2)_x$, and therefore $(m_1)_x = (m_2)_x$.

Now let G^* be the graph formed from $G_1^1 \cap G_2^1$ together with v_0 and the three adjacent edges. G^* satisfies $i(V^*) = 2|V^*| - 1$, and it is not x -constructive, a contradiction.

Case (2) Since G_1^1 is $(2, 2)$ -tight and G_2^2 is $(2, 3)$ tight, the intersection $G_1^1 \cap G_2^2$ satisfies

$$i(V_1^1 \cap V_2^2) = 2|V_1^1 \cap V_2^2| - \ell, \text{ where } \ell \in \{2, 3\}.$$

If $\ell = 2$, then $V_1^1 \cap V_2^2$ is critical but G_2^2 is $(2, 3)$ -tight, a contradiction.

Therefore $\ell = 3$, and $G_1^1 \cap G_2^2$ is connected by Lemma 5.2 (3). All paths from v_1 to v_2 have x -gain $(m_0 + m_1)_x$, and net gain $m_0 + m_2$. As in the previous case, $(m_1)_x = (m_2)_x$.

Now consider the union $V_1^1 \cup V_2^2$, which is critical. Let G^* be the $(2, 1)$ -tight graph created by adding the vertex v_0 and its three adjacent edges to $G_1^1 \cup G_2^2$. As a subgraph of G , the gain assignment on G^* must be x -constructive.

But all paths from v_1 to v_2 have x -gain $(m_0 + m_1)_x = (m_0 + m_2)_x$, and the gain assignment on G^* is not x -constructive, a contradiction.

Case (3) $G_1^2 \cap G_2^2$ satisfies

$$i(V_1^2 \cap V_2^2) = 2|V_1^2 \cap V_2^2| - \ell, \text{ where } \ell \in \{2, 3, 4\}.$$

By the same argument as before $\ell \neq 2$.

Suppose $\ell = 3$. Lemma 5.2 (3) implies the intersection $G_1^2 \cap G_2^2$ is connected, and all paths in $G_1^2 \cap G_2^2$ have net gain $m_0 + m_1 = m_0 + m_2$, which implies that $m_1 = m_2$. Letting G^* be the graph created from $G_1^2 \cap G_2^2$ by adding the vertex v_0 and its three adjacent edges, we find that G^* is a $(2, 2)$ -tight subgraph of G which is not constructive, a contradiction.

Thus $\ell = 4$. When $|V_1^2 \cap V_2^2| > 2$, the intersection is connected, and we again obtain $m_1 = m_2$. As in Case (2), considering the graph $G_1^2 \cup G_2^2$ provides a contradiction.

Finally we consider the case when $|V_1^2 \cap V_2^2| = 2$, in which case the intersection $E_1^2 \cap E_2^2$ is empty. This corresponds to the bunny ears class of graphs. Then the graph G^* formed from adding v_0 and its three adjacent edges to $G_1^2 \cap G_2^2$ is $(2, 1)$ -tight, and moreover it must be the *minimal* $(2, 1)$ -tight subgraph (it does not contain any $(2, 1)$ -tight subgraphs). Therefore, there are no loops in G (since loops always form the minimal $(2, 1)$ -tight subgraph), and we may add the loop edge $\{v_2, v_2; m_1 - m_2\}$. \square

Now observe that if at most one of v_1, v_2 lies in the circuit (and hence v_0 does not) then we may proceed exactly as for the fixed torus. This is clearly the case if v_1, v_2 are not in the circuit so suppose v_1 is. There are two cases, (1) there are two copies of $v_1 v_0$ and (2) there are two copies of $v_2 v_0$. In (1) the inverse Henneberg 2c operation creates a subgraph which is not $(2, 1)$ -sparse (the circuit with a loop added to v_1) and in (2) the inverse Henneberg 2c operation creates a second circuit contrary to uniqueness. Hence no possible inverse Henneberg operation alters the existing circuit.

6.3 Three Distinct Neighbours

Proposition 6.4. *Let $\langle G, m \rangle$ be periodic orbit framework where G is a $P(2, 1)$ graph, and m is \mathcal{T}_x -constructive. Let v_0 be a three-valent circuit vertex adjacent to three distinct vertices v_1, v_2, v_3 , all of which are in the circuit. Let the three edges adjacent to v_0 be given by $\{v_0, v_i; m_i\}$. Then v_0 is admissible.*

Proof. We proceed in a similar fashion to the proof of Proposition 6.3. The three candidate edges are

$$\{v_1, v_2; m_2 - m_1\}, \{v_2, v_3; m_3 - m_2\}, \{v_3, v_1; m_1 - m_3\}.$$

Suppose we cannot add any of these. We will consider four distinct cases. First we describe some notation.

If we cannot add the edge $\{v_1, v_2; m_2 - m_1\}$, then we must have one of two scenarios:

- There is a subgraph $G_1^1 \subset G$ which contains v_1, v_2 but not v_0, v_3 , such that V_1^1 is critical that will not have an x -constructive gain assignment with the addition of the candidate edge $\{v_1, v_2; m_2 - m_1\}$. That is, G_1^1 is $(2, 2)$ -tight, contains no x -constructive cycles, and all paths from v_1 to v_2 have x -gain $(m_2 - m_1)_x$. Note that G_1^1 cannot contain a $(2, 1)$ subgraph, since that subgraph would be x -constructive by hypothesis. Furthermore, G_1^1 must be constructive.
- There is a $(2, 3)$ -tight subgraph $G_1^2 \subset G$ which contains v_1, v_2 but not v_0, v_3 , that will not have a constructive gain assignment with the addition of $\{v_1, v_2; m_2 - m_1\}$. That is, G_1^2 contains no constructive cycles, and all paths from v_1 to v_2 have net gain $m_0 + m_1$.

We remark that any $(2, 1)$ -tight subgraph containing v_1 and v_2 must be the whole circuit, so this situation does not arise.

Similarly, if we cannot add the edge $\{v_2, v_3; m_3 - m_2\}$, then we have the two scenarios:

- G_2^1 is a $(2, 2)$ -tight subgraph containing v_2, v_3 but not v_0, v_1 , and with all paths from v_2 to v_3 having x -gain $(m_3 - m_2)_x$.
- G_2^2 is a $(2, 3)$ -tight subgraph containing v_2, v_3 but not v_0, v_1 , and with all paths from v_2 to v_3 having net gain $m_3 - m_2$.

Finally if we cannot add the edge $\{v_3, v_1; m_1 - m_3\}$, then we have the analogous subgraphs

- G_3^1 is a $(2, 2)$ -tight subgraph containing v_1, v_3 but not v_0, v_2 , and with all paths from v_3 to v_1 having x -gain $(m_1 - m_3)_x$.
- G_3^2 is a $(2, 3)$ -tight subgraph containing v_1, v_3 but not v_0, v_2 , and with all paths from v_3 to v_1 having net gain $m_1 - m_3$.

We consider triples of subgraphs which prevent the addition of any edge, and we have four cases.

- (1) G_1^1, G_2^1, G_3^1
- (2) G_1^1, G_2^1, G_3^2
- (3) G_1^1, G_2^2, G_3^2
- (4) G_1^2, G_2^2, G_3^2

The other combinations follow by symmetry. In fact we will show that we can treat cases (1) and (2) together.

Cases (1), (2)

Consider the intersection of G_1^1 and G_2^1 . We show that these two subgraphs cannot co-exist, which eliminates these first two cases.

First note that $i(V_1^1 \cap V_2^1) \leq 2|V_1^1 \cap V_2^1| - 2$ as a subgraph of the $(2, 2)$ -tight graphs G_1^1, G_2^1 . In addition $|E_1^1 \cup E_2^1| \leq 2|V_1^1 \cup V_2^1| - 2$, since $v_1, v_2, v_3 \in V_1^1 \cup V_2^1$, and the addition of v_0 cannot create an overbraced subgraph. Together these facts mean that we have equality in both cases. In addition, the intersection graph is connected.

Now consider the graph $G_1^1 \cup G_2^1$. Lemma 6.2 implies all paths from v_1 to v_3 have x -gain $(m_3 - m_1)_x$.

We also claim that $G_1^1 \cup G_2^2$ contains no x -constructive cycles. This can be proved using a similar argument.

Continuing with Cases (1) and (2), let G^* be the graph formed by appending the vertex v_0 and its three adjacent edges to $G_1^1 \cap G_2^1$. Since $G_1^1 \cap G_2^1$ is $(2, 2)$ -tight, G^* is a $(2, 1)$ -tight subgraph of G . As such, it must be x -constructive. But $G_1^1 \cap G_2^1$ does not contain any x -constructive cycles, and adding v_0 does not create any new constructive cycles, by our claim. This is a contradiction, and therefore G_1^1 and G_2^1 cannot both exist.

Case (3) Consider the intersection $G_1^1 \cap G_2^2$. Lemma 5.3 (2) gives two cases. *Case A.* $G_1^1 \cup G_2^2$ is $(2, 2)$ -tight, and $G_1^1 \cap G_2^2$ is $(2, 3)$ -tight (and hence connected).

We may now argue along the same lines as Cases (1) and (2). That is, $G_1^1 \cup G_2^2$ is not x -constructive, and nor is the graph G^* formed by appending v_0 and its three adjacent edges to $G_1^1 \cup G_2^2$, which is the contradiction, since G^* is a $(2, 1)$ -tight subgraph of G .

Case B. $G_1^1 \cup G_2^2$ is semi-critical. Let $G_4 = G_1^1 \cup G_2^2$ and consider $G_4 \cap G_3^2$. Since V_4 is semi-critical, and G_3^2 is $(2, 3)$ -tight, we know that

$$i(V_4 \cap V_3^2) + i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| + 2|V_4 \cap V_3^2| - 6. \quad (7)$$

Since $v_1, v_2, v_3 \in V_4 \cup V_3^2$, but v_0 is not, it must be the case that

$$2|V_4 \cup V_3^2| - 4 \leq i(V_4 \cup V_3^2) \leq 2|V_4 \cup V_3^2| - 2.$$

Hence there are 3 cases to analyse.

Case I) If we have equality, $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 2$, then

$$i(V_4 \cap V_3^2) = 2|V_4 \cap V_3^2| - 4 \quad (8)$$

by (7). In this case, the induced subgraph on the vertices $V_4 \cup V_3^2$ contains no more than $i(V_4 \cup V_3^2)$ edges, since otherwise we would have an overbraced subgraph.

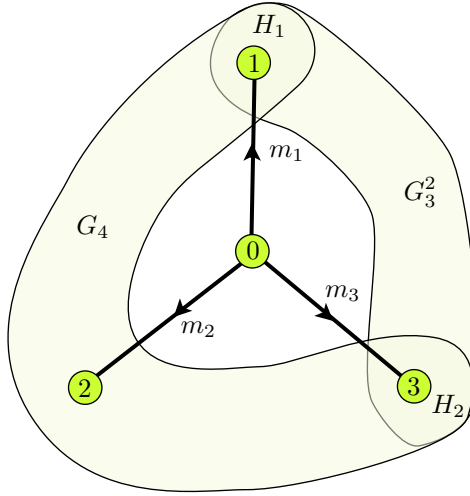


Figure 8: Proof of Proposition 6.4, Case 3,B,I. $G_4 = G_1^1 \cup G_2^2$, and the intersection $G_4 \cap G_3^2$ is disconnected into at least two components, including H_1 and H_2 . In Case 3,B,II, the intersection is connected, in which case it consists of a single edge linking H_1 and H_2 .

It follows that the intersection, $G_4 \cap G_3^2$ is disconnected, since $v_2 \notin G_4 \cap G_3^2$ (Figure 8). Say the component that lies completely within G_1^1 is called H_1 , and the component lying completely within G_2^2 is called H_2 . If $V(H_1)$ and $V(H_2)$ both contain more than one vertex, then we immediately obtain a contradiction, with the following argument. Since $|E(H_1)| \leq 2|V(H_1)| - 2$, and $|E(H_2)| \leq 2|V(H_2)| - 3$, then

$$i(V_4 \cap V_3^2) = |E(H_1)| + |E(H_2)| \leq 2(|V(H_1)| + |V(H_2)|) - 5.$$

But this contradicts (8).

Therefore, it must be the case that one or both of H_1, H_2 consists only of a single vertex. Suppose $|V(H_2)| = 1$, and therefore $|E(H_2)| = 0$. Then by (8), we find that $|E(H_1)| = 2|V(H_1)| - 2$ (and may therefore be a single vertex as well, but need not be). Note all paths from v_1 to v_3 must pass through the vertex v_2 . Thus $G_4 \cup G_3^2$ is a $(2, 2)$ -tight subgraph such that all paths from v_i to v_j have gain $m_j - m_i$ for $1 \leq i < j \leq 3$. Let G^* be the graph formed from $G_4 \cap G_3^2$ by adding the vertex v_0 and its three adjacent edges. Then G^* has $i(V^*) = 2|V^*| - 1$, but no x -constructive cycles. This is a contradiction, since we assumed that G has no such subgraphs.

Case II) If it is the case that $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 3$, then

$$i(V_4 \cap V_3^2) = 2|V_4 \cap V_3^2| - 3 \quad (9)$$

by (7). The intersection, $G_4 \cap G_3^2$ must be connected, since otherwise we would have overbraced subgraphs of G_1^1 or G_2^2 . In particular, there is exactly one edge, say e , which connects $H_1 \subset G_1^1 \cap G_3^2$ with $H_2 \subset G_2^2 \cap G_3^2$ (see Figure 8). Moreover, since $|E(H_1)| \leq 2|V(H_1)| - 2$, it follows that H_2 consists of a single vertex, namely v_3 . H_1 is then $(2, 2)$ -tight.

We claim that the graph $G_4 \cup \{e\}$ is a $(2, 2)$ -tight subgraph of G which contains no x -constructive cycles, and all paths from v_i to v_j have gain $m_j - m_i$ for $1 \leq i < j \leq 3$. These facts follow from similar arguments to those used in Cases 1 and 2. For example, any path from v_1 to v_2 that does not lie completely within G_1^1 must pass through the new edge e . But then e joins $H_1 \subset G_1^1$ at some vertex u which is also contained within G_3^2 . Since H_1 is connected, there is a path from u to v_1 within H_1 . The path $v_3 \rightarrow u \rightarrow v_1$ lies within G_3^2 , and therefore the net gain on this path is $m_1 - m_3$. The path from v_2 to v_3 is therefore:

$$\begin{array}{c} v_2 \rightarrow u \rightarrow v_3 \\ \underbrace{v_2 \rightarrow u \rightarrow v_1}_{(m_1 - m_2)_x} \underbrace{\rightarrow u \rightarrow v_3}_{(m_3 - m_1)_x}, \end{array}$$

and hence the net x -gain from v_2 to v_3 on any path that goes through the edge e is $(m_3 - m_2)_x$, which proves (a). Similar arguments apply to show (b) and (c).

Case III) If it is the case that $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 4$, then

$$i(V_4 \cap V_3^2) = 2|V_4 \cap V_3^2| - 2 \quad (10)$$

by (7). As before we find that $H_1 \subset G_1^1 \cap G_3^2$ is $(2, 2)$ -tight, and $H_2 \subset G_2^2 \cap G_3^2$ consists of a single vertex. In addition, $G_4 \cap G_3^2$ contains two additional edges, say e and f , as a consequence of (9). But then $G_4 \cup \{e, f\}$ is a $(2, 1)$ -tight subgraph containing v_1, v_2, v_3 but not v_0 , a contradiction.

Case (4)

Consider the intersection of G_1^2 and G_2^2 . From the previous arguments, we have the following cases:

- A. $|V_1^2 \cap V_2^2| > 1$, and $i(V_1^2 \cap V_2^2) \leq 2|V_1^2 \cap V_2^2| - 3$, or
 B. $|V_1^2 \cap V_2^2| = 1$, and $i(V_1^2 \cap V_2^2) = 0$.

Case A. We know that $G_1^2 \cap G_2^2$ has

$$i(V_1^2 \cap E_2^2) \leq 2|V_1^2 \cap V_2^2| - 3, \text{ and}$$

$$i(V_1^2 \cup E_2^2) \geq 2|V_1^2 \cup V_2^2| - 3.$$

However, since the union does not contain any constructive cycles (as the union of two graphs without constructive cycles by the arguments of Cases 1 and 2), in fact these inequalities are both tight.

Let G^* be the graph formed from $G_1^2 \cup G_2^2$ together with v_0 and its three adjacent edges. Then $|E^*| = 2|V^*| - 2$, and must therefore be constructive as a subgraph of G . But, by the arguments of Cases (1) – (2), we find that all paths from v_1 to v_3 have net gain $m_3 - m_1$, and therefore, G^* contains no constructive cycles, a contradiction.

Case B. Now suppose that $|V_1^2 \cap V_2^2| = 1$ (i.e. $V_1^2 \cap V_2^2 = \{v_2\}$), and therefore $i(V_1^2 \cup V_2^2) = 2|V_1^2 \cup V_2^2| - 4$.

Let $G_4 = G_1^2 \cup G_2^2$ and consider $G_4 \cap G_3^2$. Since G_4 has $|E_4| = 2|V_4| - 4$, and G_3^2 is $(2, 3)$ -tight, we know that

$$i(V_4 \cap V_3^2) + i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| + 2|V_4 \cap V_3^2| - 7. \quad (11)$$

Since $v_1, v_2, v_3 \in V_4 \cup V_3^2$, but v_0 is not, it must be the case that

$$i(V_4 \cup V_3^2) \leq 2|V_4 \cup V_3^2| - 2.$$

Here we again have three cases, depending on the number of edges in $G_4 \cup G_3^2$.

Case I) If $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 2$, then $i(V_4 \cap V_3^2) \leq 2|V_4 \cap V_3^2| - 5$. The intersection is therefore disconnected, and must consist of a single vertex $H_1 \subset G_1^2$, where $H_1 = \{v_1\}$, and a $(2, 3)$ -tight subgraph $H_2 \subset G_3^2$ (i.e. H_2 is not a singleton). But then $G_4 \cup G_3^2$ is a $(2, 2)$ -tight subgraph of G with no constructive cycles, a contradiction.

Case II) If $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 3$, then $i(V_4 \cap V_3^2) \leq 2|V_4 \cap V_3^2| - 4$. The intersection may be connected or disconnected, with $H_1 \subset G_1^2$, and $H_2 \subset G_3^2$ being either single vertices, or $(2, 3)$ -tight subgraphs. In any case, let G^* be the subgraph of G consisting of $G_4 \cup G_3^2$ together with v_0 and its three adjacent edges. Then G^* is a $(2, 2)$ -tight subgraph of G with no constructive cycles, a contradiction.

Case III) If $i(V_4 \cup V_3^2) = 2|V_4 \cup V_3^2| - 4$, then $i(V_4 \cap V_3^2) \leq 2|V_4 \cap V_3^2| - 3$. The intersection must be connected, and therefore $E(V_4 \cap V_3^2)$ contains more edges than $E_4 \cap E_3^2$. There are at most two induced edges, otherwise G_4 together with the three induced edges would form a $(2, 1)$ -tight subgraph containing v_1, v_2, v_3 , a contradiction.

If there are exactly two induced edges, then G_4 together with the induced edges $\{e, f\}$ is a $(2, 2)$ -tight subgraph of G . By arguments similar to those in Case 3, this is a subgraph with no constructive cycles, a contradiction.

If there is exactly one induced edge, then G_4 together with the induced edge $\{e\}$ is a $(2, 3)$ -tight subgraph with no constructive cycles, and all paths from v_i to v_j have gain $m_j - m_i$ for $1 \leq i < j \leq 3$. Letting G^* be the graph formed from $G_4 \cup \{e\}$ by adding v_0 and its three adjacent edges. Then G^* is a $(2, 2)$ -tight subgraph of G with no constructive cycles, a contradiction. \square

When two of the vertices lie in the circuit:

Proposition 6.5. *Let $\langle G, m \rangle$ be periodic orbit framework where G is a $P(2, 1)$ graph, and m is \mathcal{T}_x -constructive. Let v_0 be a three-valent vertex adjacent to three distinct vertices v_1, v_2, v_3 , where the edges adjacent to v_0 are $\{v_0, v_i; m_i\}$. Suppose that v_1, v_2 are circuit vertices, but v_0 and v_3 lie outside the circuit. Then v_0 is admissible.*

Proof. The three candidate edges are

$$\{v_1, v_2; m_2 - m_1\}, \{v_2, v_3; m_3 - m_2\}, \{v_3, v_1; m_1 - m_3\}.$$

Suppose G_1^0 is a $(2, 1)$ -tight subgraph of G containing v_1, v_2 , but not v_0 or v_3 . If G_2^1 or G_3^1 exist we contradict Lemma 5.4 (2). Similarly if G_2^2 exists and intersects the circuit in more than one vertex we contradict Lemma 5.4 (3). Finally if $|V_1^0 \cap V_2^2| = 1$ then we contradict Lemma 5.6.

In other words, if two of the vertices v_1, v_2, v_3 lie in the circuit (or any $(2, 1)$ -tight subgraph containing the circuit), then at least one of the other two candidate edges may always be added. \square

Proposition 6.6. *Let $\langle G, m \rangle$ be periodic orbit framework where $G = (V, E)$ is a $P(2, 1)$ graph with unique over-critical set X , and m is \mathcal{T}_x -constructive. Let $v_0 \in V$ have $N(v_0) = \{v_1, v_2, v_3\}$ and suppose $|X \cap N(v_0)| \leq 1$. Then v_0 is admissible.*

Proof. Lemma 5.5 reduces this to a fixed torus problem, thus the result follows from Theorem 2.3. \square

We now reach the stated goal of this section.

Proof of Theorem 6.1. One direction follows from the definitions of the gain-preserving Henneberg operations. The converse follows from Propositions 6.3, 6.4, 6.5 and 6.6. \square

7 Extensions

Up to this point, we have focused on frameworks which are on a variable torus, where the variability is in the x -direction only. It is immediate that we can apply this result to frameworks which are variable in the y -direction only, simply by demanding that our framework be \mathcal{T}_y -constructive. We now outline several other variations on the torus with one degree of freedom.

7.1 Fixed Area and Angle

One variation of the variable torus is the torus whose area remains fixed, as does the angle between the two generators, which we denote \mathcal{T}_{vol}^2 . It is generated by the lattice matrix

$$L_{vol}(t) = \begin{pmatrix} x(t) & 0 \\ 0 & kx(t) \end{pmatrix}.$$

In this case the angle is constrained to remain fixed at $\pi/2$.

Certainly, if a framework is generically rigid on \mathcal{T}_x^2 or \mathcal{T}_y^2 , then it is generically rigid on \mathcal{T}_{vol}^2 too. Of course, the natural way to understand the fixed area setting is to consider frameworks on the torus with two degrees of freedom, which we do not address here.

7.2 Flexible Angle

Let \mathcal{T}_θ^2 be the torus generated by the lattice matrix

$$L_\theta = \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \end{pmatrix}.$$

This is the torus that has generators with fixed lengths, and a variable angle between them. Then the rigidity matrix for frameworks on \mathcal{T}_θ^2 has a column corresponding to the variable $\theta(t)$. Instead of requiring an x -constructive cycle, the necessary condition for rigidity can be seen to be that the critical subgraph contains a cycle with net gain (m_1, m_2) satisfying $m_1 m_2 \neq 0$. The other requirements for rigidity are as for frameworks on \mathcal{T}_x^2 .

7.3 Frameworks on a Variable Cylinder

As a direct consequence of Theorem 1.2 we obtain a characterisation of the generic rigidity of frameworks which are periodic in one direction only. That is, Theorem 1.2 provides necessary and sufficient conditions for the rigidity of frameworks on the cylinder with variable circumference, where the cylinder is a “flat cylinder.” That is, we are *not* considering frameworks supported on surfaces in three dimensional space as in [17]. See Figure 9 for an example. Such frameworks are similar to *frieze patterns*, although we assume that such patterns exhibit *only* translational symmetry (in one direction), and do not have any of the other symmetries of frieze patterns.



(b)

Let $\langle G, m \rangle$ be a gain graph with gain assignments from \mathbb{Z} . Let $\hat{m} : E \rightarrow \mathbb{Z}^2$ be the gain assignment on G given by $\hat{m}(e) = (m(e), 0)$. Since the rigidity matrix for the cylinder and the rigidity matrix for \mathcal{T}_x^2 are identical (each has exactly one lattice column), we have the following proposition.

Proposition 7.1. *A periodic orbit graph $\langle G, m \rangle$ is generically rigid on the variable cylinder \mathcal{C} if and only if $\langle G, \hat{m} \rangle$ is generically rigid on \mathcal{T}_x^2 .*

8.1 Fully Variable Torus

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and the fact that the variants of V -replacement do not necessarily preserve the relevant counting conditions.

8.2 Global Periodic Rigidity

The global rigidity of frameworks in Euclidean space is well studied. As in the case of rigidity there is a celebrated combinatorial characterisation in the plane, see [11], but no such characterisation is known in higher dimensions. The result in the plane relied on inductive constructions of circuits in the plane rigidity matroid [2] and crucially of 3-connected, redundantly rigid graphs [11]. It is tempting then, as a first step, to take the inductive constructions provided here and in [20] and try to provide related constructions for the circuits in the fixed (or flexible) torus rigidity matroid. As far as we know this has not yet been addressed.

8.3 Periodic Body-Bar Frameworks

Recently, a characterisation of the generic rigidity of periodic body-bar frameworks on the fixed torus has been developed, when $d \leq 3$ [21]. The body-bar setting is somewhat different from the present study, since we are no longer working within a combinatorial subclass of the full inductive class (as we are with $P(2, 1)$ graphs in the class of all $(2, 1)$ -tight graphs). For this reason, it is possible to use existing inductive characterisations of the relevant combinatorial structures [5]. However, it may be possible to extend those results to the partially variable torus, which may require a more subtle approach, as in the present paper.

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